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Adding first mS and then T to both sides of the equation we find

$$(1 + m)S = m(S + T), \qquad S + T = (1 + m)T,$$

hence

$$S = \frac{m}{1 + m}(S + T), \qquad T = \frac{1}{1 + m}(S + T).$$

Now obviously $S + T = xy - x_0y_0 = p(x^{m+1} - x_0^{m+1})$, so that we may write

$$S = \frac{mp}{1 + m}(x^{m+1} - x_0^{m+1}), \qquad T = \frac{p}{1 + m}(x^{m+1} - x_0^{m+1}).$$

If the foregoing proof seems to lack rigor, the objection may be easily removed as follows. We have shown that $\Delta S/\Delta T = m + hF^2$ and $\Delta T = phx_r^m$, hence

$$\Delta S = m\cdot\Delta T + hF^2\cdot\Delta T = m\cdot\Delta T + ph^2F^2x_r^m,$$

$$\Sigma\Delta S = m\cdot\Sigma\Delta T + ph^2\Sigma(F^2x_r^m).$$

In the limit $\Sigma\Delta S = S$, $\Sigma\Delta T = T$, hence in the limit

$$S = mT + \lim\,[ph^2\Sigma(F^2x_r^m)],$$

and it remains to show that the second term on the right is 0.

Let G^2 represent the greatest of the n values of $F^2x_r^m$, then $\Sigma(F^2x_r^m) < n\cdot G^2$, and $ph^2\Sigma(F^2x_r^m) < ph^2nG^2 = ph(x - x_0)G^2$, since $hn = x - x_0$, and therefore

$$\lim\,[ph^2\Sigma(F^2x_r^m)] = \lim\,[ph(x - x_0)G^2] = p(x - x_0)G^2\cdot\lim h = 0.$$

In the case of the common parabola, $m = 2$, $F^2 = 1/x_r$, and $G^2 = x$. In that case, too, the foregoing proof holds for any segment of the parabola. For if the line through the middle point of the chord and parallel to the axis of the parabola be chosen for the y-axis, and the tangent parallel to the chord for the x-axis, the equation of the parabola remains unchanged, and areas of the parallelograms corresponding to ΔS and ΔT are equal to $\Delta S\cdot\sin\theta$ and $\Delta T\cdot\sin\theta$ respectively, θ being the angle between the coördinate axes. The ratio of these areas remains therefore unchanged and the conclusion regarding the ratio of S to T remains valid.

II. Geometrical Construction of the Roots of a Cubic, and Inscription of a Regular Heptagon in a Circle.

By C. B. Haldeman, Ross, Butler County, Ohio.

1. The equation

$$q(x^2 - y^2) + 2x\sqrt{-(r^2 + q^3)} + 2ry = 0 \tag{1}$$

represents a real equilateral hyperbola when q^3 is negative and numerically greater than r^2, and

$$x^2 + y^2 = -4q \tag{2}$$

is an equation of a circle, which will be real when q is negative. Eliminating x from these equations, we have

$$(y^3 + 3qy + 2r)(qy - 2r) = 0.$$

The first of these factors is a form to which the general cubic equation may be reduced. From this it appears that the three real roots of a cubic equation, reduced to the above form, may be represented by the ordinates of the intersections of an equilateral hyperbola and a circle. The three intersections, whose ordinates are the three real roots of this equation, are the vertices of an equilateral triangle; because the roots are

$$y = -2\sqrt{-q}\sin\tfrac{1}{3}\sin^{-1}\frac{r}{q\sqrt{-q}},$$

$$y = 2\sqrt{-q}\sin\tfrac{1}{3}\left(\pi + \sin^{-1}\frac{r}{q\sqrt{-q}}\right),$$

$$y = -2\sqrt{-q}\sin\tfrac{1}{3}\left(\pi - \sin^{-1}\frac{r}{q\sqrt{-q}}\right);$$

since $y = -2S\sqrt{-q}$ substituted in $y^3 + 3qy + 2r = 0$ gives $3S - 4S^3 = \dfrac{r}{q\sqrt{-q}}$; and comparing this with

$$3\sin A - 4\sin^3 A = \sin 3A,$$

which is well known, we obtain the values of y.

2. The equation $z^3 + Rz^2\sqrt{7} - R^3\sqrt{7} = 0$ gives the value of z which represents the side of a regular heptagon inscribed in a circle whose radius is R. The transformation

$$z = y - \frac{R\sqrt{7}}{3}$$

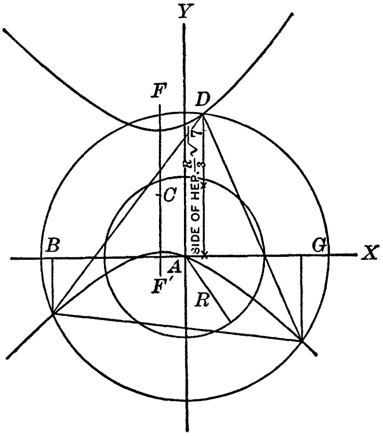
will reduce this equation to

$$y^3 - \frac{7R^2y}{3} - \frac{13R^3\sqrt{7}}{27} = 0.$$

Comparing this with the equation

$$y^3 + 3qy + 2r = 0,$$

we have $q = -7R^2/9$, $r = -13R^3\sqrt{7}/54$; also we have the radius of (2) equal to $2R\sqrt{7}/3$.



With A as a center and $2R\sqrt{7}/3$ as a radius describe a circle BGD and with A as the origin of coördinates, after substituting the values of q and r just found in (1), draw the equilateral hyperbola

$$x^2 - y^2 + \frac{R\sqrt{21}}{7} x + \frac{13R\sqrt{7}}{21} y = 0$$

with the diameter BG prolonged as the axis of X and AY as the axis of Y. Then will the ordinate of intersection at D minus $R\sqrt{7}/3$ be the side of a regular heptagon inscribed in a circle whose radius is R; that is,

$$z = \frac{R\sqrt{7}}{3} [2 \sin \tfrac{1}{3}(\pi + \sin^{-1} \tfrac{13}{14}) - 1]$$

is the side of the regular heptagon required.

III. On the Summation of Certain Series.

By O. Schmiedel, Parsons College.

The summation of a finite number of terms of series such as

$$1 + 3x + 6x^2 + 10x^3 + \cdots,$$

$$1 + \cos\theta + \cos 2\theta + \cos 3\theta + \cdots,$$

$$1 + x\cos\theta + x^2\cos 2\theta + x^3\cos 3\theta + \cdots,$$

$$1 + 3x\cos\theta + 6x^2\cos 2\theta + 10x^3\cos 3\theta + \cdots,$$

and the evaluation of the complete series is a problem of frequent occurrence, and is easily solved by the use of De Moivre's Theorem;[1] but greater insight to the student into their nature and mutual relation is had by direct application of the more common processes of analysis.

It is proposed to produce a formula sufficiently comprehensive to embrace all series of the types suggested, including as particular cases the binomial formula for negative integral exponents and the theorem that the difference of the same powers is divisible by the difference of the first.

Let the second of the series above be deduced first, for illustration.

Assuming the identity

$$2 \sin\theta/2 \cos t\theta = \sin(t + 1/2)\theta - \sin(t - 1/2)\theta,$$

assigning to t values from 0 to n, and adding the results, gives directly the expression for the sum of the series of cosines, namely:

$$(1) \qquad \sum_{t=0}^{t=n} \cos t\theta = 1/2 + 1/2\,\frac{\sin(n + 1/2)\theta}{\sin\theta/2} = \frac{1}{\sin\theta/2} \sin\frac{n+1}{2}\theta \cos\frac{n}{2}\theta.$$

In like manner is found the sum of sines:

$$(2) \qquad \sum_{t=0}^{t=n} \sin t\theta = \frac{1}{\sin\theta/2} \sin\frac{n+1}{2}\theta \sin\frac{n}{2}\theta;$$

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[1] Chrystal, *Algebra*, Vol. II, 2 ed., p. 273.